This freedom could also be used to design an adaptive scheme for the choice of the exponential family S. In this respect, it would also be useful to obtain for all $t \geq 0$ an estimate of the distance between the optimal filter density p_t and the PF density p_t^π, in terms of the total residual norm history $\{r_s^*, 0 \leq s \leq t\}$.

Finally, we would like to define PF's for discrete-time systems and relate this problem with the work of Kulhavý [15] and [16]. Another motivation for this study will be to obtain efficient numerical schemes for the solution of the stochastic differential equation satisfied by the PF parameters, i.e., (6) for a general family S, or (10) for the family $S_\bullet$.

Each of these problems requires further investigation and will be addressed in subsequent works.

REFERENCES

[1] B. Hanzon, "A differential-geometric approach to approximate nonlinear filtering," in *Geometrization of Statistical Theory,* C. T. J. Dodson, Ed. Univ. Lancaster: ULMD Pub., 1987, pp. 219–223.

[2] D. Brigo, "On the nice behavior of the Gaussian projection filter with small observation noise," *Syst. Contr. Lett.,* vol. 26, no. 5, pp. 363–370, 1995.

[3] ——, "New results on the Gaussian projection filter with small observation noise," *Syst. Contr. Lett.,* vol. 28, no. 5, pp. 273–279, 1996.

[4] D. Brigo, B. Hanzon, and F. LeGland, "A differential geometric approach to nonlinear filtering: The projection filter," Internal Memo. 914, IRISA, June 1995; available at URL: ftp://ftp.irisa.fr/techreports/1995/PI-954.ps.gz.

[5] ——, "On the relationship between assumed density filters and projection filters," Tinbergen Inst., Feb. 1996, discussion paper TI 7-96-18.

[6] ——, "The exponential projection filter and the selection of the exponential family," in *Proc. 2nd Portuguese Conf. Automat. Contr.,* APCA, Porto, Sept. 1996, pp. 251–256.

[7] ——, "A differential geometric approach to nonlinear filtering: The projection filter," in *Proc. 34th IEEE-CSS Conf. Decision Contr.,* New Orleans, LA, Dec. 1995, pp. 4006–4011.

[8] D. Brigo, "Filtering by projection on the manifold of exponential densities," Ph.D. dissertation, Dept. Economics and Econometrics, Vrije Univ., Amsterdam, The Netherlands, Oct. 1996.

[9] S.-I. Amari, "Differential-geometrical methods in statistics," in *Lecture Notes in Statistics*, no. 28. Berlin: Springer-Verlag, 1985.

[10] M. H. A. Davis and S. I. Marcus, "An introduction to nonlinear filtering," in *Stochastic Systems: The Mathematics of Filtering and Identification and Applications, Les Arcs 1980,* vol. C78, NATO Advanced Study Inst. Series, M. Hazewinkel and J. C. Willems, Eds. Dordrecht: D. Reidel, 1981.

[11] B. Hanzon and R. Hut, "New results on the projection filter," in *Proc. 1st European Contr. Conf.,* Grenoble, 1991, vol. I, pp. 623–628.

[12] M. Hazewinkel, S. I. Marcus, and H. J. Sussmann, "Nonexistence of finite dimensional filters for conditional statistics of the cubic sensor problem," *Syst. Contr. Lett.,* vol. 3, no. 6, pp. 331–340, 1983.

[13] H. J. Kushner and P. Dupuis, "Numerical methods for stochastic control problems in continuous time," in *Applications of Mathematics*, vol. 24. New York: Springer-Verlag, 1992.

[14] F. LeGland, "Time discretization of nonlinear filtering equations," in *Proc. 28th IEEE–CSS Conf. Decision Contr.,* Tampa, FL, Dec. 1989, pp. 2601–2606.

[15] R. Kulhavý, "Recursive nonlinear estimation: A geometric approach," *Automatica,* vol. 26, no. 3, pp. 545–555, 1990.

[16] ——, "Recursive nonlinear estimation: Geometry of a space of posterior densities," *Automatica,* vol. 28, no. 2, pp. 313–323, 1992.

L_2-Induced Norms and Frequency Gains of Sampled-Data Sensitivity Operators

J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg

***Abstract*— This paper develops exact, computable formulas for the frequency gain and L_2-induced norm of the sensitivity operator in a sampled-data control system. With sampled data, we refer to a system that combines both continuous-time and discrete-time signals, which is studied in continuous time. The expressions are obtained using lifting techniques in the frequency domain and have application in performance and stability robustness analysis taking into account full intersample information.**

***Index Terms*— Frequency response, generalized sampled-data holds, L_2-induced norms, robustness, sampled-data systems, sensitivity analysis.**

I. INTRODUCTION

This paper studies the computation of the L_2-induced norm of the sensitivity operator in a sampled-data (SD) control system. The term SD indicates that we approach the system in continuous time, i.e., considering full intersample information. The L_2-induced norm is the operator norm when inputs and outputs live in the space of square-integrable signals L_2. For linear time-invariant (LTI) systems, the L_2-induced operator norm is the H_∞ norm of the system transfer matrix.

The concepts and methods associated with LTI H_∞ control bear no immediate extension to SD systems since, when intersample behavior is taken into account, the operators are time-varying and no transfer functions are associated with them. In view of this, considerable research during recent years has focused on the study of L_2-norms and H_∞-related problems for SD systems.

Early works addressing the computation of L_2-induced norms of SD systems include [1]–[4]. References [1]–[3] considered simple open-loop connections involving a sampler and a zero-order hold (ZOH). A formula for the computation of the L_2-induced norm in a feedback setup was given in [4] under the assumptions of band-limited input signals.

More general results on the computation of L_2-induced norms and the first solutions to SD H_∞-optimal control problems were obtained using different time-domain frameworks. One of these frameworks is the so-called *lifting technique* (see [5]–[11]). Time-domain approaches that dispense with the use of lifting techniques include [12]–[15]. In particular, Sun *et al.* [14] applied techniques based on *linear systems with jumps* to the synthesis of H_∞ controllers for SD systems. An interesting conclusion from [14] is that if one does *not* assume the hold device to be a ZOH, then the optimal solution involves the use of a generalized SD hold function (GSHF) *à la* Kabamba [16].

A recently introduced concept, which is closely related to L_2-induced norms, is that of the *frequency gain* of an SD operator.

Manuscript received June 20, 1996; revised October 29, 1996.

J. H. Braslavsky was with the Department of Electrical and Computer Engineering, The University of Newcastle, Australia. He is now with the ECE Department, University of California, Santa Barbara, CA 93105 USA (e-mail: julio@seidel.ece.ucsb.edu).

R. H. Middleton is with the Department of Electrical and Computer Engineering, The University of Newcastle, Australia.

J. S. Freudenberg is with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109 USA.

Publisher Item Identifier S 0018-9286(98)00963-5.

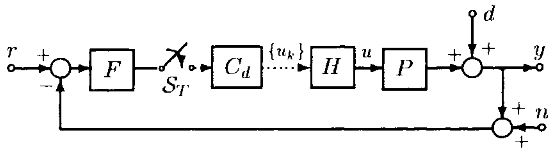

Fig. 1. Sampled-data control system.

This concept extends the LTI notion of frequency response to SD systems, in the sense that the maximum magnitude of the frequency gain of an SD operator is its L_2-induced norm [17]–[20]. Yamamoto and Khargonekar [17] used lifting techniques to compute the frequency gain of a general SD system, while Hagiwara *et al.* [18] obtained similar results for the class of SD *compact* operators using a frequency-domain framework developed in [21]. Relations between both approaches have been discussed in [19]. Although more general, the procedures proposed in [17] do not seem to have an easy numerical implementation. On the other hand, the formulas provided in [18] are readily numerically implementable in a reliable fashion. An iterative procedure was also suggested in this latter paper for the computation of the frequency gain of operators such as the sensitivity operator, which, as it turns out, is *not* compact.

In this paper, we apply a frequency-domain lifting technique to obtain exact formulas for the computation of the frequency gain of the SD sensitivity operator. The L_2-induced norm is then obtained from the frequency gain by performing a simple search of a maximum over a finite interval of frequencies. These expressions have a direct application in performance and stability robustness analyses of SD systems. In particular, our results are formulated in terms of GSHF's, thus allowing the analysis of design techniques using D/A devices other than the ZOH.

Notation and setup are defined in Section II. In Section III, we review the frequency-domain lifting formalism that will be used to state and prove our results, which appear in Section IV. Expressions for the numerical implementation of the results and an illustrative example are given in Section V.

II. Preliminaries

We consider the multivariable SD feedback system shown in Fig. 1, where P and F are the transfer functions of the plant and anti-aliasing filter, C_d is the digital controller, and r, d, and n are the command, disturbance, and noise inputs to the system. The system output is y, and u and $\{u_k\}$ are the analog and discrete control inputs. The plant and controller are assumed to be proper, and the filter strictly proper and stable,[1] and they are all free of unstable hidden modes.

We denote the sampling period by T and the sampling frequency by $\omega_s \triangleq 2\pi/T$. The *Nyquist frequency range* is defined as the interval $\Omega_N \triangleq [-\omega_s/2, \omega_s/2]$. If v is a continuous-time signal, we define the sampling operation with period T by $\mathcal{S}_T v = \{v_k\}_{k=-\infty}^{\infty}$, where the sequence $\{v_k\}_{k=-\infty}^{\infty}$ represents the sampled signal, with $v_k = v(kT)$ for any integer k. The z-transform operator is denoted by $\mathcal{Z}$, i.e., $\mathcal{Z}\{u_k\} \triangleq \sum_{k=-\infty}^{\infty} u_k z^{-k}$, and the Laplace transform operator is denoted by $\mathcal{L}$, $\mathcal{L}u = U$.

The hold device H is a GSHF [16], defined by

$$u(t) = h(t-kT)u_k, \qquad kT \le t < (k+1)T, \qquad k \in \mathbb{Z}.$$

The function h, which characterizes the hold, is defined over the interval $[0, T]$ and satisfies some mild technical conditions described in [23]. Associated with this hold we define its *frequency response function* by $H = \mathcal{L}h$. Since h is supported on a finite interval, it follows that H has no singularities at any finite s in $\mathbb{C}$; e.g., for the ZOH, $H(s) = (1-e^{-sT})/s$. Frequency responses for other types of hold functions are derived in [23].

[1] The assumption that the filter is strictly proper is standard and guarantees that the sampling operation is well defined (cf., [10] and [22]). The assumption of stability is only made for simplicity of exposition and may be removed.

We denote by $(FPH)_d$ the *discretized plant*, defined as

$$(FPH)_d \triangleq \mathcal{Z}\mathcal{S}_T\mathcal{L}^{-1}FPH$$

where $\mathcal{L}^{-1}FPH$ denotes the inverse Laplace transform of FPH. In connection with $(FPH)_d$, we also introduce the *discrete input sensitivity function* and the *discrete output complementary sensitivity function*, respectively, as

$$S_d \triangleq [I + C_d(FPH)_d]^{-1} \quad \text{and} \quad T_d \triangleq (FPH)_d S_d C_d. \tag{1}$$

In [23], the well-known nonpathological sampling condition for plants discretized with a ZOH is generalized to the case of a GSHF. This result was also extended to the multivariable case in [24]. In particular, since for GSHF's H may have zeros in the right half-plane, it is necessary to require that none of these coincides with an unstable pole of the analog plant. Under the nonpathological sampling hypothesis, it is straightforward to extend the exponential and L_2 input–output stability results of [25] and [26] to the case of a GSHF. We shall assume throughout that the system of Fig. 1 is L_2 input–output stable.

The assumptions on P, H, and F stated above suffice to guarantee [27] that $(FPH)_d$ satisfies the well-known formula

$$(FPH)_d(e^{j\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_k(j\omega)P_k(j\omega)F_k(j\omega) \tag{2}$$

where the notation $Y_k(s)$ represents $Y(s + jk\omega_s)$, with k an entire number. This notation will be frequently used.

We shall also use the function G, defined as

$$G(j\omega) \triangleq \frac{1}{T}P(j\omega)H(j\omega)S_d(e^{j\omega T})C_d(e^{j\omega T}). \tag{3}$$

Associated with G and F, we introduce two discrete transfer matrices required to formulate our results

$$G_d(e^{j\omega T}) \triangleq \sum_{k=-\infty}^{\infty} G_k^*(j\omega)G_k(j\omega) \tag{4}$$

and

$$F_d(e^{j\omega T}) \triangleq \sum_{k=-\infty}^{\infty} F_k(j\omega)F_k^*(j\omega) \tag{5}$$

where F^* denotes the conjugated transpose of F. Note that if y, n, and d are valued in $\mathbb{R}^m$, then G_d and F_d are $m \times m$ discrete transfer matrices.

A characteristic feature of SD systems is evident from (2), namely the response of the discretized plant at a frequency $\omega \in \Omega_N$ depends upon the response of the analog plant, prefilter, and hold function at an infinite number of frequencies. Indeed, it is well known that the steady-state response of a stable SD system to a sinusoidal input consists of a fundamental component and infinitely many aliases shifted by multiples of the sampling frequency. Analogous expressions are obtained for the response to more general inputs, such as noise n and output disturbance d (cf., [21], [28], and [29]). In particular, if n is in $L_2(0, \infty)$ and N is its Laplace transform, then we have that the system response is given by

$$Y(j\omega) = -P(j\omega)H(j\omega)S_d(e^{j\omega T})C_d(e^{j\omega T})\,(FN)_d(e^{j\omega T}) \tag{6}$$

where $(FN)_d(e^{j\omega T}) = 1/T\sum_{k=-\infty}^{\infty} F_k(j\omega)N_k(j\omega)$. Similarly, for the response to a disturbance d in $L_2(0, \infty)$, we have that

$$Y(j\omega) = D(j\omega) - P(j\omega)H(j\omega)S_d(e^{j\omega T}) \cdot C_d(e^{j\omega T})\,(FD)_d(e^{j\omega T}). \tag{7}$$

Although an SD system is time-varying, its intrinsic periodic nature allows the use of model transformation techniques that yield time-invariant characterizations. The following section deals with one of them.

III. Frequency-Domain Lifting

Several frameworks are available for the treatment of SD systems embodying intersample information in the model. Among them, we have time-domain approaches, as the lifting technique of, e.g., [5], [7], [17], and frequency-domain approaches, as the *FR-operators* introduced in [21]. We use a frequency-domain setting that we refer to as a *frequency-domain* lifting technique. The transformation involved in this approach may be viewed as a generalization of the Fourier transform in the FR-operators framework of [21]. The idea of lifting in frequency domain is not new; it was developed in the signal processing literature for linear discrete-time periodic systems; see for example [30].

Let y be a signal in the space $L_2(0, \infty)$. Then, it is a fact that its Fourier transform $Y(j\omega)$ belongs to $L_2(-\infty, \infty)$. Now, from $Y(j\omega)$ construct the sequence of functions $\{Y_k(j\omega)\}_k = \{Y(j(\omega+k\omega_s))\}_k$, for ω in the Nyquist range Ω_N and k integer. Arrange this sequence in an infinite vector, which we denote by

$$\mathbf{y}(\omega) \triangleq [\cdots, Y_1^{\mathrm{t}}(j\omega), Y_0^{\mathrm{t}}(j\omega), Y_{-1}^{\mathrm{t}}(j\omega), \cdots]^{\mathrm{t}} \tag{8}$$

where the superscript "t" denotes transpose. We say that the infinite-dimensional vector $\mathbf{y}$ is the *lifting* of Y, and we denote the lifting operation as $\mathbf{y} = \mathcal{F}Y$. As a function, $\mathbf{y}$ is defined at almost every ω in Ω_N and takes values in ℓ_2. Moreover, these ℓ_2-valued functions form a Hilbert space [31] under the norm and inner product

$$\|\mathbf{y}\| \triangleq \left(\int_{\Omega_N} \|\mathbf{y}(\omega)\|_{\ell_2}^2 \, d\omega\right)^{1/2}$$
$$\langle \mathbf{y}, \mathbf{x}\rangle \triangleq \int_{\Omega_N} \langle \mathbf{y}(\omega), \mathbf{x}(\omega)\rangle_{\ell_2} \, d\omega.$$

We denote this space by $L_2(\Omega_N; \ell_2)$. Since the elements of $L_2(\Omega_N; \ell_2)$ are essentially rearrangements of those of $L_2(-\infty, \infty)$, both spaces are in fact isomorphic with preservation of norm [19].

A key point of the lifting is that in the new space, operators of the SD system are represented as *multiplication operators* described by infinite-dimensional "transfer matrices." In other words, if $\mathcal{M}$ is a bounded operator in L_2, and $\mathcal{F}\mathcal{M}\mathcal{F}^{-1}$ is the corresponding operator in $L_2(\Omega_N; \ell_2)$, then its action can be described as $(\mathcal{F}\mathcal{M}\mathcal{F}^{-1}\mathbf{y})(\omega) = \mathbf{M}(\omega)\mathbf{y}(\omega)$. An important consequence of these facts is that the L_2-induced norm of the operator can then be computed as [6], [19]

$$\|\mathcal{M}\| = \sup_{\omega \in \Omega_N} \|\mathbf{M}(\omega)\| \tag{9}$$

where the supremum is understood as the essential supremum in Ω_N, and $\|\mathbf{M}(\omega)\|$ is the ℓ_2-induced operator norm of $\mathbf{M}(\omega)$. The scalar-valued function $\|\mathbf{M}\|\colon \Omega_N \to \mathbb{R}_0^+$ is the so-called *frequency gain* of the SD operator $\mathcal{M}$ [17], [18]. Notice the similarity of (9) to the expression of the L_2-induced norm of an LTI operator, i.e., the H_∞-norm of a transfer matrix.

IV. L_2-Induced Norms of Sensitivity Operators

We concentrate on the sensitivity and complementary sensitivity operators in the SD system of Fig. 1. These operators are defined as the mappings relating output disturbance d and noise n to the output y and are, respectively, denoted by

$$\mathcal{S}\colon \begin{array}{ccc} L_2 & \to & L_2 \\ \mathcal{S}d & \mapsto & y \end{array} \quad \text{and} \quad \mathcal{T}\colon \begin{array}{ccc} L_2 & \to & L_2 \\ \mathcal{T}n & \mapsto & y. \end{array}$$

Under the assumptions of closed-loop L_2-stability, $\mathcal{S}$ and $\mathcal{T}$ are bounded operators on L_2.

The actions of the complementary sensitivity and sensitivity operators are respectively defined in frequency domain by the steady-state responses (6) and (7). These equations are alternatively written as

$$\mathbf{y} = -\mathbf{T}\mathbf{n} \quad \text{and} \quad \mathbf{y} = \mathbf{S}\mathbf{d} \tag{10}$$

after applying the lifting of Section III. Here, $\mathbf{T}(\omega)$ is the following infinite-dimensional transfer matrix defined on Ω_N (cf., [19] and [21]):

$$\mathbf{T} = \begin{bmatrix} \ddots & \vdots & \vdots & \\ \cdots & G_kF_k & G_kF_{k-1} & \cdots \\ \cdots & G_{k-1}F_k & G_{k-1}F_{k-1} & \cdots \\ & \vdots & \vdots & \ddots \end{bmatrix} \tag{11}$$

where F is the transfer matrix of the prefilter and G is the function introduced in (3). The corresponding matrix for $\mathcal{S}$ is thus given by $\mathbf{S} = \mathbf{I} - \mathbf{T}$, where $\mathbf{I}$ is the identity operator on ℓ_2.

Operators $\mathbf{T}$ and $\mathbf{S}$ are infinite-dimensional transfer matrix representations of the SD complementary sensitivity and sensitivity operators $\mathcal{T}$ and $\mathcal{S}$. We are interested in the computation of their frequency gains $\|\mathbf{T}(\omega)\|$ and $\|\mathbf{S}(\omega)\|$. The corresponding L_2-induced norms are then obtained, from (9), by searching for suprema over the finite interval Ω_N.

An important fact about the complementary sensitivity operator $\mathcal{T}$ is that it has finite rank (and, therefore, is compact).

Lemma IV.1: If the inputs to the system in Fig. 1 are valued in $\mathbb{R}^m$, then $\mathcal{T}$ has at most rank m.

Proof: Partition $F(j\omega)$ by rows, and $G(j\omega)$ by columns, i.e., $F = [f_1^*, f_2^*, \cdots, f_m^*]^*$, and $G = [g_1, g_2, \cdots, g_m]$. Denote the liftings for F^* and G by $\mathbf{f} = \mathcal{F}F^*$, and $\mathbf{g} = \mathcal{F}G$. Hence, $\mathbf{f} = [\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m]$, and $\mathbf{g} = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m]$, where each column $\mathbf{f}_i = \mathcal{F}f_i^*$ in $\mathbf{f}$ and $\mathbf{g}_i = \mathcal{F}g_i$ in $\mathbf{g}$ lives in $L_2(\Omega_N; \ell_2)$ because F and G are both stable and strictly proper. Then, the action of $\mathbf{T}$ can be alternatively written as

$$\mathbf{T}\mathbf{n} = \sum_{i=1}^{m} \mathbf{g}_i \langle \mathbf{n}, \mathbf{f}_i\rangle_{\ell_2} \tag{12}$$

where, $\langle \mathbf{n}, \mathbf{f}_i\rangle_{\ell_2}$ is a scalar-valued function defined a.e. on Ω_N. From (12), $\mathbf{T}$ is the sum of m rank-one operators on $L_2(\Omega_N; \ell_2)$; it thus has at most rank m, and so does $\mathcal{T}$. ■

Since $\mathcal{T}$ is compact, the numerical computation of the norm of $\mathcal{T}$ may be approximated by truncating $\mathbf{T}$ between harmonics $-n$ and n, say, and evaluating the maximum singular value of the $(2n+1) \times (2n+1)$ matrix so obtained [21]. The convergence of the approximation could be slow, though, since in general $G(j\omega)$ and $F(j\omega)$ decay as $1/\omega^p$, for some integer p depending on the relative degrees of the transfer matrices involved.

In fact, because $\mathcal{T}$ is finite-rank, more efficient ways of computing $\|\mathbf{T}(\omega)\|$ are possible. In [18], the authors show that $\|\mathbf{T}(\omega)\|$ is given as the maximum eigenvalue $\lambda_{\max}[\cdot]$ of an associated finite-dimensional discrete transfer matrix evaluated on the unit circle. We quote this result for convenience, and we provide an alternative proof that uses geometric arguments.

Theorem IV.2 (L_2-Induced Norm of $\mathcal{T}$): If the SD system of Fig. 1 is L_2-input–output stable, then

$$\|\mathcal{T}\| = \sup_{\omega \in \Omega_N} \|\mathbf{T}(\omega)\| \tag{13}$$

where

$$\|\mathbf{T}(\omega)\|^2 = \lambda_{\max}\left[G_d(e^{j\omega T})F_d(e^{j\omega T})\right] \tag{14}$$

and G_d and F_d are as defined in (4) and (5).

Proof: From (9), we have that $\|\mathcal{T}\| = \sup_{\omega \in \Omega_N} \|\mathbf{T}(\omega)\|$. Fix ω in Ω_N. Since $\mathbf{T}$ is a finite-rank operator in ℓ_2, from (12) we can write as a dyadic product $\mathbf{T} = \mathbf{g}\mathbf{f}^*$. We can then decompose ℓ_2 into $\ell_2 = \wp_F \oplus \wp_F^{\perp}$, where $\wp_F$ is the subspace of ℓ_2 spanned by the range of $\mathbf{f}$ and $\wp_F^{\perp}$ its orthogonal complement. Hence, if $\mathbf{v}$ is a vector in $\wp_F^{\perp}$ then $\mathbf{T}\mathbf{v} = 0$, and we can write

$$\begin{aligned}\|\mathbf{T}\| &= \sup_{\substack{v \in \ell_2 \\ v \neq 0}} \frac{\|\mathbf{T}\mathbf{v}\|_{\ell_2}}{\|\mathbf{v}\|_{\ell_2}} \\ &= \sup_{\substack{v \in \wp_F \\ v \neq 0}} \frac{\|\mathbf{T}\mathbf{v}\|_{\ell_2}}{\|\mathbf{v}\|_{\ell_2}}.\end{aligned}$$

Since vectors of ℓ_2 in $\wp_F$ can be finitely parameterized as $\mathbf{v} = \mathbf{f}\alpha$, where α belongs to $\mathbb{C}^m$, with m the number of inputs of F, we then have that

$$\begin{aligned}\|\mathbf{T}\|^2 &= \sup_{\substack{\alpha \\ \mathbf{f}\alpha \neq 0}} \frac{\alpha^* \mathbf{f}^* \mathbf{f} \mathbf{g}^* \mathbf{g} \mathbf{f}^* \mathbf{f} \alpha}{\alpha^* \mathbf{f}^* \mathbf{f} \alpha} \\ &= \lambda_{\max}\Big[(\mathbf{f}^*\mathbf{f})^{1/2} (\mathbf{g}^*\mathbf{g}) (\mathbf{f}^*\mathbf{f})^{1/2}\Big]. \qquad (15)\end{aligned}$$

Note that $(\mathbf{g}^*\mathbf{g})(\omega) = G_d(e^{j\omega T})$ and $(\mathbf{f}^*\mathbf{f})(\omega) = F_d(e^{j\omega T})$ are the $m \times m$ discrete transfer matrices defined in (4) and (5). In particular, $\mathbf{f}^*\mathbf{f}$ is nonsingular because F is full column rank.

Finally, since eigenvalues are invariant under similarity transformations, (14) follows from (15), completing the proof. ■

The case of $\mathcal{S}$ has to be considered more carefully, since this is a *noncompact* operator, and as such it may not be in principle approximable by sequences of finite-rank operators, meaning that the norms of progressive truncations of $\mathbf{S}$ may not necessarily converge to the norm of the operator.

Frequency gains of possibly noncompact SD operators have been discussed in [17], but, as pointed out in [18], the proposed procedure seems in general hard to implement numerically. In [18], a numerically reliable method is suggested for the case of operators like $\mathcal{S}$, i.e., the sum of a compact and a constant operator. Yet, to compute the frequency gain $\|\mathbf{S}\|$, this last method still requires a γ-iteration *at each frequency* $\omega \in \Omega_N$.

The following theorem gives an exact, computable formula for the frequency-gain and L_2-induced norm of the SD sensitivity operator $\mathcal{S}$. Our result relies on the fact that $\mathcal{S}$ verifies the complementarity relation $\mathcal{S} = \mathcal{I} - \mathcal{T}$, and since $\mathcal{T}$ is of finite rank, it turns out that the computation of the frequency gain of $\mathcal{S}$ also reduces to a finite-dimensional eigenvalue problem. As for Theorem IV.2, these results admit a simple and reliable numerical implementation.

Theorem IV.3 (L_2-Induced Norm of $\mathcal{S}$): If the SD system of Fig. 1 is L_2-input–output stable, then

$$\|\mathcal{S}\| = \sup_{\omega \in \Omega_N} \|\mathbf{S}(\omega)\| \qquad (16)$$

where

$$\|\mathbf{S}(\omega)\|^2 = 1 + \lambda_{\max}\begin{bmatrix} F_d(e^{j\omega T})G_d(e^{j\omega T}) - T_d(e^{j\omega T}) & -F_d(e^{j\omega T}) \\ T_d^*(e^{j\omega T})G_d(e^{j\omega T}) - G_d(e^{j\omega T}) & -T_d^*(e^{j\omega T}) \end{bmatrix}$$

G_d, F_d are the functions given by (4) and (5), and T_d is the discrete complementary sensitivity function defined in (1).

Proof: The same idea for the proof of Theorem IV.2 works here. Again, for a fixed ω in Ω_N, decompose ℓ_2 into $\ell_2 = \wp_{(F,G)} \oplus \wp_{(F,G)}^{\perp}$, where $\wp_{(F,G)}$ denotes the subspace spanned by both $\mathbf{f}$ and $\mathbf{g}$, and $\wp_{(F,G)}^{\perp}$ its orthogonal complement. Since $\mathbf{S}$ is block diagonal in these spaces, we have that

$$\begin{aligned}\|\mathbf{S}\| &= \max\left\{ \sup_{\substack{v \in \wp_{(F,G)} \\ v \neq 0}} \frac{\|\mathbf{S}\mathbf{v}\|_{\ell_2}}{\|\mathbf{v}\|_{\ell_2}},\ \sup_{\substack{v \in \wp_{(F,G)}^{\perp} \\ v \neq 0}} \frac{\|\mathbf{S}\mathbf{v}\|_{\ell_2}}{\|\mathbf{v}\|_{\ell_2}} \right\} \\ &= \max\left\{ \sup_{\substack{v \in \wp_{(F,G)} \\ v \neq 0}} \frac{\|\mathbf{S}\mathbf{v}\|_{\ell_2}}{\|\mathbf{v}\|_{\ell_2}},\quad 1 \right\}. \qquad (17)\end{aligned}$$

Now, any vector $\mathbf{v}$ in $\wp_{(F,G)}$ can be finitely parameterized as

$$\mathbf{v} = [\mathbf{f}, \mathbf{g}]\gamma \qquad (18)$$

with γ in C^{2m}. Denote $\mathbf{h} \triangleq [\mathbf{f}, \mathbf{g}]$, and $M \triangleq \mathbf{h}^*\mathbf{h}$. Notice that M is a finite-dimensional, positive semidefinite Hermitian matrix. Using the notation introduced in (4) and (5), and noticing that the discrete complementary sensitivity in (1) may be expressed as $T_d = \mathbf{f}^*\mathbf{g}$, we can write M as

$$M = \begin{bmatrix} F_d & T_d \\ T_d^* & G_d \end{bmatrix}.$$

Introducing the matrix N

$$N \triangleq \begin{bmatrix} G_d & -I \\ -I & 0 \end{bmatrix}$$

it then follows that $\mathbf{h}^*\mathbf{S}^*\mathbf{S}\mathbf{h} = \mathbf{h}^*(\mathbf{I} - \mathbf{f}\mathbf{g}^*)(\mathbf{I} - \mathbf{g}\mathbf{f}^*)\mathbf{h} = (I + MN)M$, and hence, we obtain from (18) that

$$\begin{aligned}\sup_{\substack{v \in \wp_{(F,G)} \\ v \neq 0}} \frac{\|\mathbf{S}\mathbf{v}\|_{\ell_2}^2}{\|\mathbf{v}\|_{\ell_2}^2} &= \sup_{\gamma \in \mathbb{C}^{2m}} \frac{\gamma^* M \gamma + \gamma^* M N M \gamma}{\gamma^* M \gamma} \\ &= 1 + \lambda_{\max}\Big[M^{1/2} N M^{1/2}\Big] \qquad (19) \\ &= 1 + \lambda_{\max}[MN]. \qquad (20)\end{aligned}$$

Since in (20) the product MN is

$$MN = \begin{bmatrix} F_d G_d - T_d & -F_d \\ T_d^* G_d - G_d & -T_d^* \end{bmatrix}$$

from (17) and (20) we see that it only remains to show that $\lambda_{\max}[MN]$ is nonnegative to complete the proof. But this follows easily from the fact that $M \geq 0$. Indeed, if $M > 0$ then

$$\delta = \begin{bmatrix} F_d & T_d \\ T_d^* & G_d \end{bmatrix}^{-1/2} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

gives $\delta^* M^{1/2} N M^{1/2} \delta = G_d \geq 0$, and thus $\lambda_{\max}$ in (19) is nonnegative. Otherwise M is necessarily singular, and thus zero must be in the spectrum of $M^{1/2} N M^{1/2}$, which shows that $\lambda_{\max}[MN] \geq 0$. The proof is now complete. ■

The closed-form expressions given by Theorems IV.2 and (16) can be used for performance and stability robustness analysis of SD systems [10], [22], [32].

In the particular case of single-input–single-output (SISO) systems, these formulas simplify and show some interesting connections. In this case, the operator $\mathcal{T}$ is then of rank one, and so the frequency gains are given by the magnitude of the frequency response of scalar discrete transfer functions.

Corollary IV.4: If the SD system of Fig. 1 is SISO, then

$$\|\mathbf{T}\| = \Phi_d |T_d| \tag{21}$$

and

$$\|\mathbf{S}\| = \tfrac{1}{2}\left(\sqrt{(\Phi_d^2 - 1)|T_d|^2 + (|S_d| + 1)^2} + \sqrt{(\Phi_d^2 - 1)|T_d|^2 + (|S_d| - 1)^2}\right) \tag{22}$$

where S_d and T_d are the discrete sensitivity and complementary sensitivity functions, evaluated at $z = e^{j\omega T}$, and

$$\Phi_d^2(e^{j\omega T}) = \frac{\left(\sum_{k=-\infty}^{\infty} |F_k(j\omega)|^2\right)\left(\sum_{k=-\infty}^{\infty} |P_k(j\omega)H_k(j\omega)|^2\right)}{|(FPH)_d(e^{j\omega T})|^2}. \tag{23}$$

Proof: The proof of (21) follows immediately from Theorem IV.2. Formula (22) may be checked by computing $\lambda_{\max}$ in (16) and after some straightforward but tedious algebraic manipulation. ■

The expressions given in Corollary IV.4 show a direct connection to the discrete sensitivity functions S_d and T_d. In fact, the magnitudes of their frequency responses are correspondingly lower bounds on the frequency gains of $\mathcal{S}$ and $\mathcal{T}$, as we see in the following corollary.

Corollary IV.5: Under the assumptions of Corollary IV.4

$$\|\mathbf{T}(\omega)\| \geq |T_d(e^{j\omega T})| \tag{24}$$

and

$$\|\mathbf{S}(\omega)\| \geq \max\{|S_d(e^{j\omega T})|, 1\} \tag{25}$$

at all frequencies ω in Ω_N.

Proof: First notice from (23) that Φ_d is greater than or equal to one at any ω in Ω_N, since by Cauchy–Schwarz

$$\begin{aligned} &|(FPH)_d(e^{j\omega T})|^2 \\ &= \left|\frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(j\omega)P_k(j\omega)H_k(j\omega)\right|^2 \\ &\leq \left(\sum_{k=-\infty}^{\infty} |F_k(j\omega)|^2\right)\left(\frac{1}{T^2}\sum_{k=-\infty}^{\infty} |P_k(j\omega)H_k(j\omega)|^2\right). \end{aligned}$$

Hence, (24) follows immediately. For (25), we have from (22) that

$$\|\mathbf{S}\| \geq \frac{||S_d| - 1| + |S_d| + 1}{2} \tag{26}$$

since $\Phi_d \geq 1$. Therefore, from (26), if $|S_d| > 1$ then $\|\mathbf{S}\| \geq |S_d|$, and otherwise $\|\mathbf{S}\| \geq 1$, which completes the proof. ■

Not surprisingly, it then follows from Corollary IV.5 that the L_2-induced norms of the discretized system also give lower bounds for the L_2-induced norms of the SD system. The gap in norms, then, is due to the intersample information missing in the discrete description of the system. Note that in this sense, $\sup_{\omega\in\Omega_N} \Phi_d$ may be interpreted as a "fidelity index," *independent of the controller*, that quantifies how close we can expect to be the discrete and SD performances.

V. Numerical Implementation

The expressions for the frequency-gains and L_2-induced norms obtained in the last section can be readily numerically implemented by computing G_d and F_d from (4) and (5). These computations can be approached as "special discretizations" by considering relations similar to (2). In this way, the arguments of $\sup_{\omega\in\Omega_N}$ in (13) and (16) are expressed by two rational transfer functions in $z = e^{j\omega T}$—the frequency gains of the SD sensitivity operators. The induced norms can then be computed by a straightforward search of maxima over the finite interval Ω_N. Similar formulas have been derived for the case of ZOH in [4, Th. 3].

A. Computation of $F_d(e^{j\omega T})$

Consider $F_d = T\mathcal{Z}\mathcal{S}_T\mathcal{L}^{-1}(F\tilde{F})$, where $\tilde{F}(s) \triangleq F(-s)^{\mathrm{t}}$, i.e., the transpose of F at $-s$. Since F is strictly proper, then the sampling of the output of $F\tilde{F}$ is well defined. If $\{a, b, c, 0\}$ is a minimal state-space realization for F, then, a minimal realization for $F\tilde{F}$ is given by

$$A = \begin{bmatrix} a & bb^{\mathrm{t}} \\ 0 & -a^{\mathrm{t}} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ -c^{\mathrm{t}} \end{bmatrix}, \qquad C = [c \quad 0]$$

and it is straightforward to see that the function $F_d(e^{j\omega T})$ is then computed as $F_d(e^{j\omega T}) = TC(e^{j\omega T}I - e^{AT})^{-1}B$.

B. Computation of $G_d(e^{j\omega T})$

The case of G_d is slightly more complicated than the previous one but can be approached in a similar fashion. From (4), we can write $G_d = \frac{1}{T}C_d^* S_d^* E_d S_d C_d$, where

$$E_d \triangleq \frac{1}{T}\sum_{k=-\infty}^{\infty} H_k^* P_k^* P_k H_k. \tag{27}$$

Hence, to compute G_d we need to evaluate $E_d(e^{j\omega T})$. This is done by discretizing the cascade of the hold $\tilde{H}$, the system $P\tilde{P}$, and the hold H. Since H is proper by definition, so is the cascade, and therefore the sampling operation is again well defined. If the plant P has a minimal realization $\{a, b, c, d\}$, then a minimal realization for $\tilde{P}P$ is given by

$$A = \begin{bmatrix} a & 0 \\ c^{\mathrm{t}}c & -a^{\mathrm{t}} \end{bmatrix}, \qquad B = \begin{bmatrix} b \\ c^{\mathrm{t}}d \end{bmatrix}$$
$$C = [d^{\mathrm{t}}c \quad -b^{\mathrm{t}}], \qquad D = [d^{\mathrm{t}}d].$$

Suppose that the hold is as defined in Section II. Then, following [23], its pulse response can be described as

$$h(t) = \begin{cases} Ke^{L(T-t)}M, & \text{if } t \in [0, T) \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

for matrices K, L, and M of appropriate dimensions. Using these data, it may be checked that the function $E_d(e^{j\omega T})$ in (27) is given by $E_d(e^{j\omega T}) = C_d(e^{j\omega T}I - A_d)B_d + D_d$, where $A_d = e^{AT}$, $B_d = \int_0^T e^{A\tau}BKe^{L\tau}M\, d\tau$, and

$$\begin{aligned} C_d &= \int_0^T M^{\mathrm{t}} e^{L^{\mathrm{t}}(T-\tau)} K^{\mathrm{t}} C e^{A\tau}\, d\tau \\ D_d &= \int_0^T M^{\mathrm{t}} e^{L^{\mathrm{t}}\tau} K^{\mathrm{t}} D K e^{L\tau} M\, d\tau + \int_0^T M^{\mathrm{t}} e^{L^{\mathrm{t}}\cdot(T-\tau)} \\ &\quad \cdot K^{\mathrm{t}} C \int_0^\tau e^{A(\tau-\sigma)} BKe^{L(T-\sigma)} M\, d\sigma\, d\tau. \end{aligned}$$

Matrices B_d, C_d, and D_d in the above expressions can be easily numerically evaluated using matrix exponential formulas suggested

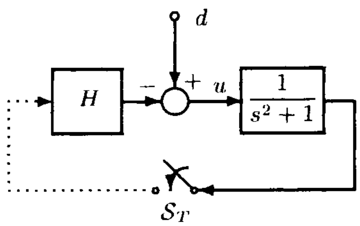

Fig. 2. System with plant input disturbance.

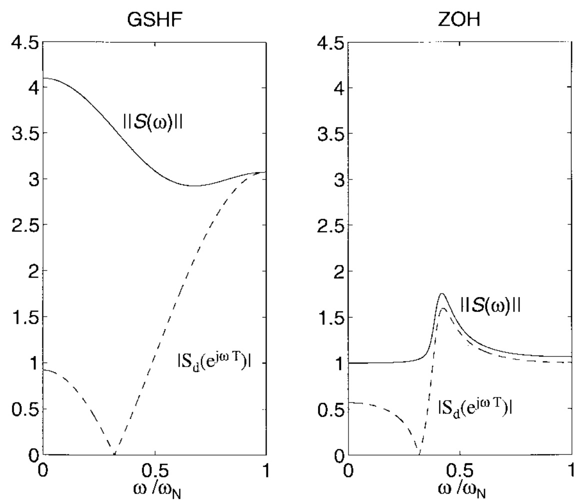

Fig. 3. Sampled-data frequency gains.

by [33] which yield

$$B_d = [e^{AT} \quad 0] \exp\left\{\begin{bmatrix} -A & BK \\ 0 & L \end{bmatrix} T\right\}\begin{bmatrix} 0 \\ M \end{bmatrix}$$
$$C_d = [M^{\mathrm{t}} \quad 0] \exp\left\{\begin{bmatrix} L^{\mathrm{t}} & K^{\mathrm{t}}C \\ 0 & A \end{bmatrix} T\right\}\begin{bmatrix} 0 \\ I \end{bmatrix}$$
$$D_d = [M^{\mathrm{t}} e^{L^{\mathrm{t}}T} \quad 0] \exp\left\{\begin{bmatrix} -L^{\mathrm{t}} & K^{\mathrm{t}}DK \\ 0 & L \end{bmatrix} T\right\}\begin{bmatrix} 0 \\ M \end{bmatrix} + [M^{\mathrm{t}} \quad 0] \exp\left\{\begin{bmatrix} L^{\mathrm{t}} & K^{\mathrm{t}}C & 0 \\ 0 & A & BK \\ 0 & 0 & -L \end{bmatrix} T\right\}\begin{bmatrix} 0 \\ e^{LT}M \end{bmatrix}.$$

C. Example: Sensitivity of GSHF Control

These formulas may be readily used to analyze sensitivity and robustness properties of SD systems. As an illustration, we compute the frequency gain of a system taken from [16, Example 1]. In this example, the author considered the problem of controlling a harmonic oscillator via GSHF in the feedback configuration of Fig. 2. A GSHF defined by

$$K = [0 \quad -1], \qquad L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} -13.1682 \\ 7.0898 \end{bmatrix}$$

and sampling period of $T = 1$ renders the closed-loop system stable with a response that is deadbeat in two sampling periods at most. As noted in [16], this system can also be asymptotically stabilized with a ZOH and a constant gain p, but then the discrete closed-loop eigenvalues cannot be arbitrarily assigned as is the case with the GSHF.

However, the GSHF solution is more sensitive and less robust than the ZOH solution. Indeed, consider the SD operator $\mathcal{S}$ mapping input disturbance d to u in Fig. 2. Fig. 3 shows the frequency gains of $\mathcal{S}$ for the GSHF solution (left) and the ZOH with $p = 0.75$ (right). For comparison, we also plotted the magnitude of the frequency responses of the corresponding discrete sensitivity functions S_d. We see from Fig. 3 that in the GSHF solution, $\|\mathbf{S}\|$ has peaks that more than double those in the ZOH solution. This indicates higher sensitivity to input disturbances and poorer robustness properties to plant perturbations.

VI. Conclusions

This paper has considered the worst-case disturbance/noise performance (L_2-induced norm) of an SD system with full intersample information. Using a frequency-domain lifting technique, we have derived exact expressions for the computation of the frequency-gain and L_2-induced norm of the SD sensitivity operator. The formulas involve a finite-dimensional eigenvalue problem that is readily numerically implemented.

These formulas have immediate application in the analysis of stability robustness for LTV unstructured perturbations and H_∞-control synthesis problems. Particularly since our expressions allow the use of GSHF's, they provide a reliable computational tool for the evaluation of performance of a general class of SD designs.

Acknowledgment

J. H. Braslavsky is indebted to Dr. M. Laca and Dr. G. Willis from the Department of Mathematics, The University of Newcastle, for valuable discussions concerning Section III.

References

[1] P. Thompson, G. Stein, and M. Athans, "Conic sectors for sampled-data feedback systems," *Syst. Contr. Lett.,* vol. 3, pp. 77–82, 1983.
[2] P. Thompson, R. Daily, and J. Doyle, "New conic sectors for sampled-data feedback systems," *Syst. Contr. Lett.,* vol. 7, pp. 395–404, 1986.
[3] T. Chen and B. Francis, "On the L_2-induced norm of a sampled-data system," *Syst. Contr. Lett.,* vol. 15, pp. 211–219, 1990.
[4] G. Leung, T. Perry, and B. Francis, "Performance analysis of sampled data control systems," *Automatica,* vol. 27, no. 4, pp. 699–704, 1991.
[5] Y. Yamamoto, "New approach to sampled-data systems: A function space method," in *Proc. 29th Conf. Decision Contr.,* 1990, pp. 1882–1887.
[6] B. Bamieh, J. Pearson, B. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems with applications to sampled-data control," *Syst. Contr. Lett.,* vol. 17, pp. 79–88, 1991.
[7] B. Bamieh and J. Pearson, "A general framework for linear periodic systems with applications to H^∞ sampled-data control," *IEEE Trans. Automat. Contr.,* vol. 37, pp. 418–435, 1992.
[8] H. Toivonen, "Sampled-data control of continuous-time systems with an H_∞ optimality criterion," *Automatica,* vol. 28, no. 1, pp. 45–54, 1992.
[9] Y. Yamamoto, "On the state space and frequency domain characterization of H^∞-norm of sampled-data systems," *Syst. Contr. Lett.,* vol. 21, pp. 163–172, 1993.
[10] G. Dullerud and K. Glover, "Robust stabilization of sampled-data systems to structured LTI perturbations," *IEEE Trans. Automat. Contr.,* vol. 38, pp. 1497–1508, 1993.
[11] Y. Yamamoto, "A function space approach to sampled data control systems and tracking problems," *IEEE Trans. Automat. Contr.* vol. 39, pp. 703–713, 1994.
[12] G. Tadmor, "$\mathcal{H}_\infty$ optimal sampled data control in continuous time systems," *Int. J. Contr.,* vol. 56, p. 99, 1992.
[13] P. Kabamba and S. Hara, "Worst-case analysis and design of sampled-data control systems," *IEEE Trans. Automat. Contr.,* vol. 38, pp. 1337–1357, 1993.
[14] W. Sun, K. Nagpal, and P. Khargonekar, "$\mathcal{H}_\infty$ control and filtering for sampled-data systems," *IEEE Trans. Automat. Contr.,* vol. 38, pp. 1162–1175, 1993.
[15] N. Sivashankar and P. Khargonekar, "Characterization of the L_2-induced norm for linear systems with jumps with applications to sampled-data systems," *SIAM J. Contr. Optim.,* vol. 32, pp. 1128–1150, 1994.
[16] P. Kabamba, "Control of linear systems using generalized sampled-data hold functions," *IEEE Trans. Automat. Contr.,* pp. 772–783, 1987.

[17] Y. Yamamoto and P. Khargonekar, "Frequency response of sampled-data systems," in *Proc. 32nd Conf. Decision Contr.*, 1993, pp. 799–804; also *IEEE Trans. Automat. Contr.*, vol. 39, pp. 166–176, 1996.
[18] T. Hagiwara, Y. Ito, and M. Araki, "Computation of the frequency response gains and H_∞-norm of a sampled-data system," *Syst. Contr. Lett.*, vol. 25, pp. 281–288, 1995.
[19] Y. Yamamoto and M. Araki, "Frequency responses for sampled-data systems—Their equivalence and relationships," *Linear Algebra Its Appl.*, vol. 206, pp. 1319–1339, 1994.
[20] T. Chen and B. Francis, *Optimal Sampled-Data Control Systems.* New York: Springer-Verlag, 1995.
[21] M. Araki, Y. Ito, and T. Hagiwara, "Frequency response of sampled-data systems," in *Proc. 12th IFAC World Congr.*, 1993, pp. VII–289, 1993; also *Automatica*, vol. 32, no. 4, pp. 483–497, 1996.
[22] N. Sivashankar and P. Khargonekar, "Robust stability and performance analysis for sampled-data systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 58–69, 1993.
[23] R. Middleton and J. Freudenberg, "Non-pathological sampling for generalized sampled-data hold functions," *Automatica*, vol. 31, 1995.
[24] R. Middleton and J. Xie, "Non-pathological sampling for high order generalized sampled-data hold functions," in *Proc. Automat. Contr. Conf.*, 1995.
[25] T. Chen and B. Francis, "Input–output stability of sampled-data systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 50–58, 1991.
[26] B. Francis and T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 820–832, 1988.
[27] J. Braslavsky, G. Meinsma, R. Middleton, and J. Freudenberg, "On a key sampling formula relating the Laplace and Z-transforms," *Syst. Contr. Lett.*, vol. 29, pp. 181–190, 1997.
[28] G. Goodwin and M. Salgado, "Frequency domain sensitivity functions for continuous time systems under sampled data control," *Automatica*, vol. 30, p. 1263, 1994.
[29] J. Freudenberg, R. Middleton, and J. Braslavsky, "Inherent design limitations for linear sampled-data feedback systems," in *Proc. Automat. Contr. Conf.*, June 1994, pp. 3227–3231; also *Int. J. Contr.*, vol. 61, no. 6, pp. 1387–1421, 1995.
[30] R. Shenoy, D. Burnside, and T. Parks, "Linear periodic systems and multirate filter design," *IEEE Trans. Signal Process.*, vol. 42, pp. 2242–2256, 1994.
[31] A. Balakrishnan, *Applied Functional Analysis.* New York: Springer-Verlag, 1981.
[32] T. Hagiwara and M. Araki, "Robust stability of sampled-data systems under possibly unstable additive/multiplicative perturbations," in *Proc. 1995 Automat. Contr. Conf.*, 1995, pp. 3893–3898.
[33] C. Van Loan, "Computing integrals involving the matrix exponential," *IEEE Trans. Automat. Contr.*, vol. 23, pp. 395–404, 1978.

Integrator Backstepping for Bounded Controls and Control Rates

Randy Freeman and Laurent Praly

***Abstract*— We present a backstepping procedure for the design of globally stabilizing state feedback control laws such that the magnitudes of the control signals and their derivatives are bounded by constants which do not depend on the initial conditions. We accomplish this by propagating such boundedness properties through each step of the recursive design.**

I. Introduction

Recursive Lyapunov design procedures developed in recent years have expanded the classes of nonlinear systems for which systematic controller designs are possible. A prime example of such a procedure is integrator backstepping (see [2] and the references therein). The flexibilities of this procedure create opportunities for the improvement of performance and the satisfaction of design constraints.

In this paper we present a new version of the backstepping procedure in which the boundedness of the control signal and its derivative are propagated through each step of the recursive design. We thereby add the powerful backstepping method to the collection of tools available for the global design of control systems with actuator constraints (see [4] and [7] for instance). The achieved bound on the control signal in our design cannot generally be made to satisfy an arbitrary prescribed constraint, unlike the bounds in the designs of [4] and [7]. However, our method applies to a much broader class of nonlinear systems, including those which do not admit controllers satisfying arbitrary constraints.

The key feature of our method is a new choice for the Lyapunov function at each step of the recursive design, a choice based on combining design flexibilities proposed in [1] and [3]. We will give our main result in Section II, followed by its proof in Section III.

II. Backstepping with Actuator Constraints

A. Main Result

Given continuous functions $f, g\colon \mathbb{R}^n \to \mathbb{R}^n$ and $h\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $h(0, 0) = 0$, we consider the single-input system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = F(x, y) + G(x, y)u \tag{1}$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ is the state variable, $u \in \mathbb{R}$ is the control variable, and F and G are given by

$$F(x, y) := \begin{bmatrix} f(x) + g(x)y \\ h(x, y) \end{bmatrix}, \qquad G(x, y) := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2}$$

Our goal in this paper is to present a set of conditions guaranteeing the existence of a stabilizing control law for (1) such that the magnitudes of both the control signal u and its derivative $\dot{u}$ are bounded by a constant which does not depend on initial conditions. Roughly

Manuscript received April 8, 1996; revised November 22, 1996.
R. Freeman is with the Department of Electrical and Computer Engineering, Northwestern University, Evanston, IL 60208-3118 USA (e-mail: freeman@ece.nwu.edu).
L. Praly is with CAS Ecole des Mines, 77305 Fontainebleau Cedex, France.
Publisher Item Identifier S 0018-9286(98)00967-2.